ON THE STABILITY OF MOTION IN THE GENERALIZED PROBLEM OF TWO IMMOVABLE CENTERS

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One of the methods for investigating the motion of the earth's artificial satellites is the approximation of the earth's gravitational potential by a potential sufficiently close to the potential of the earth. At the same time the approximating potential is chosen such that the problem could be solved in quadratures (expansion of the potential into a series of Legendre polynomials [1,2], potential of two immovable centers [3]). Then the qualitative analysis of the motion is also simplified.

E.P. Aksenov, E.A. Grebenikov and V.G. Demin believe that to date the most general of all the considered approximating potentials of the earth is the potential of two complex conjugate masses located at a certain complex distance from each other. With kind permission of Aksenov, Grebenikov and Demin, the author has utilized the formulation of the "generalized" problem of two immovable centers for investigating the stability of certain types of orbits for this problem.

1. Formulation of the problem. Let us choose a rectangular system of coordinates $Oxyz$ with the origin at the center of mass of the attracting points M_1 and M_2 such that the z-axis would lie along the line joining M_1 and M_2. The equations of motion for the points can then be written as

$$\frac{d^2x}{dt^2} = \frac{\partial U}{\partial x}, \qquad \frac{d^2y}{dt^2} = \frac{\partial U}{\partial y}, \qquad \frac{d^2z}{dt^2} = \frac{\partial U}{\partial z}$$

where the attraction potential is of the form

$$U = \frac{fM}{2}\left[\frac{1+i\sigma}{\sqrt{x^2+y^2+[z-c(\sigma+i)]^2}} + \frac{1-i\sigma}{\sqrt{x^2+y^2+[z-c(\sigma-i)]^2}}\right] \qquad (1.1)$$

If M is taken as the mass of the earth, then c and σ can be chosen so that the first three terms of the Legendre polynomial expansion of the earth potential would coincide with the potential (1.1).

Let us replace the variables x, y, z by the new variables u, v, w as follows

$$x = c\cosh v \sin u \cos w, \qquad y = c\cosh v \sin u \sin w \qquad z = c\sigma + c\sinh v \cos u \tag{1.2}$$

In this case the energy and the surface integrals will be of the form

$$T - U = h, \qquad \dot{w} \cosh^2 v \sin^2 u = c_1 \tag{1.3}$$

Here T is the kinetic energy and U the potential in the new variables

$$T = \frac{c^2}{2}\left[(\dot{u}^2 + \dot{v}^2)(\sinh^2 v + \cos^2 u) + \dot{w}^2 \cosh^2 v \sin^2 u\right], \qquad U = \frac{fM}{c}\frac{\sinh v - \sigma \cos u}{\sinh^2 v + \cos^2 u}$$

Also, let us introduce the new regulating variable τ in place of time t

$$dt = (\sinh^2 v + \cos^2 u)\, d\tau \tag{1.4}$$

Then, substituting the variables (1.2) and lowering the system order by two by the use of the first integrals (1.3), we obtain

$$\frac{d^2u}{d\tau^2} = \frac{fM\sigma}{c^3}\sin u + \frac{c_1^2 \cos u}{\sin^2 u} - \frac{h}{c^2}\sin 2u, \qquad \frac{d^2v}{d\tau^2} = \frac{fM}{c^3}\cosh v - \frac{c_1^2 \sinh v}{\cosh^3 v} + \frac{h}{c^2}\sinh 2v \tag{1.5}$$

Since the system of equations of motion is reducible to two independent equations (1.5) this then permits the investigation of stability with respect to a part of the variables. Furthermore, it can be seen from (1.4) that in investigating stability τ will have the same role as t.

2. Stability of ellipsoidal orbits. Let us augment the system of equations (1.5) with the equations

$$dh / d\tau = 0, \qquad dc_1 / d\tau = 0 \tag{2.1}$$

The system of equations, second in (1.5) and (2.1), permits a particular solution

$$v = v_0, \qquad v' = 0, \qquad h = h_0, \qquad c_1 = c_{10} \tag{2.2}$$

This solution exists if v_0 is a root of equation

$$\frac{fM}{c^3}\cosh v_0 - \frac{c_{10}^2 \sinh v_0}{\cosh^3 v_0} + \frac{h_0}{c^2}\sinh 2v_0 = 0$$

In this case the point will be located on the surface of the ellipsoid

$$\frac{x^2}{c^2\cosh^2 v_0} + \frac{y^2}{c^2\cosh^2 v_0} + \frac{(z - c\sigma)^2}{c^2\sinh^2 v_0} = 1$$

The equatorial surface of this ellipsoid coincides with the equatorial surface of the earth, the semi-axes and the eccentricity of which are

$$a = c\cosh v_0, \qquad b = c\sinh v_0, \qquad e = 1/\cosh v_0 \tag{2.3}$$

For $c_{10} = 0$ the orbit will be polar and elliptic. If it is assumed that $\sigma = 0$, then the potential (1.1) becomes

$$U = \frac{fM}{2}\left[\frac{1}{\sqrt{x^2+y^2+|z-ci|^2}} + \frac{1}{\sqrt{x^2+y^2+|z+ci|^2}}\right] \tag{2.4}$$

The stability of motion along elliptic orbits in a field with the potential (2.4) was investigated by Aksenov, Grebenikov and Demin. Since σ is not included in the system of equations (2.1) or in the second equation in (1.5), the result obtained by the above authors can be utilized.

Let us introduce the following notation for the perturbations:

$$v = v_0 + x_1, \qquad v' = x_2, \qquad h = h_0 + x_3, \qquad c_1^2 = c_{10}^2 + x_4$$

Then the differential equations for perturbed motion become

$$\frac{dx_1}{d\tau} = x_2, \qquad \frac{dx_3}{d\tau} = 0, \qquad \frac{dx_4}{d\tau} = 0$$

$$\frac{dx_2}{d\tau} = \frac{fM}{c^3}\cosh(v_0+x_1) - \frac{h_0+x_3}{c^2}\sinh 2(v_0+x_1) - \frac{(c_{10}^2+x_4)\sinh(v_0+x_1)}{\cosh^3(v_0+x_1)}$$

which possess the first integrals

$$F_1 = x_2^2 - \frac{2fM}{c^3}[\sinh(v_0+x_1) - \sinh v_0] + 2\,\frac{h_0+x_3}{c^2}\sinh^2(v_0+x_1) -$$

$$-2\frac{h_0}{c^2}\sinh^2 v_0 - \frac{c_{10}^2+x_4}{\cosh^2(v_0+x_1)} + \frac{c_{10}^2}{\cosh^2 v_0} = \text{const}, \qquad F_2 = x_3 = \text{const}, \quad F_3 = x_4 = \text{const}$$

Following Chetaev [4] we will construct the Liapunov function in the form of a combination of integrals

$$W = F_1 - 2\sinh^2 v_0 F_2 + \frac{1}{\cosh^2 v_0}F_3 + A_2F_2^2 + A_3F_3^2 =$$

$$= x_2^2 + \left(\frac{fM\cosh^2 v_0}{c^3\sinh v_0} - \frac{4c_{10}^2}{\cosh^4 v_0}\right)x_1^2 + A_2x_3^2 + A_3x_4^2 + 2\sinh 2v_0x_1x_3 + \frac{2\sinh v_0}{\cosh^3 v_0}x_1x_4 + \ldots$$

Utilizing the Silvester criterion, it is possible to obtain a sufficient condition for which the function W is positive definite, at least for the small values of x_1, x_2, x_3, x_4. This condition will be unique since the undetermined multipliers A_2 and A_3 are selected so that the remaining conditions of the Silvester criterion are fulfilled. This condition is

$$\frac{fM\cosh^2 v_0}{c^3\sinh v_0} > \frac{4c_{10}^2}{\cosh^4 v_0} \tag{2.5}$$

Since the derivative of the function W is equal to zero then, according to a Liapunov theorem [5], the motion (2.2) will be stable with respect to the semi-axes and the eccentricity of the ellipsoid if the condition (2.5) is fulfilled. By obtaining the sufficient condition for stability (2.5), Aksenov, Grebenikov and Demin conclude the stability investigation of the ellipsoidal orbits.

It is easy to show, however, that the inequality (2.5) will always be fulfilled for real earth satellites. Indeed, taking into account the notation (2.3), the inequality (2.5) can be written as

$$fM\frac{a^6}{c^6} > 4c^3c_{10}^2\frac{b}{c} \tag{2.6}$$

Since $b \leqslant a$, (2.6) will be fulfilled if

$$fMa^5 > 4c^8c_{10}^2 \tag{2.7}$$

Let **d** = **r** x **V**, where **r** and **V** are, respectively, the radius vector and the velocity vector of the point, while d_{10} is the projection of the vector **d** on the z-axis, corresponding to the initial values (2.2). Then

$$c_{10}^2 = \frac{d_{10}^2}{c^4} \leqslant \frac{d^2}{c^4} = \frac{|\mathbf{r}\times\mathbf{V}|^2}{c^4} \leqslant \frac{r^2V^2}{c^4}$$

Taking the last inequality into account and since $r \leqslant a$ it can be stated that (2.7) will be fulfilled if

$$fMa^3 > 4c^4V^2 \tag{2.8}$$

It follows from the first integral in (1.3) that

$$\frac{V^2}{2} = \frac{fM}{2}\left[\frac{1+i\sigma}{\sqrt{x^2+y^2+[z-c(\sigma+i)]^2}} + \frac{1-i\sigma}{\sqrt{x^2+y^2+[z-c(\sigma-i)]^2}}\right] + h$$

but since the motion takes place in a bounded region then $h < 0$, i.e.

$$V^2 < fM\left[\frac{1+i\sigma}{\sqrt{x^2+y^2+[z-c(\sigma+i)]^2}} + \frac{1-i\sigma}{\sqrt{x^2+y^2+[z-c(\sigma-i)]^2}}\right] \tag{2.9}$$

Expanding the right-hand side of the inequality (2.9) into a series of Legendre polynomials, one can be convinced that if (2.9) is fulfilled then $V^2 < fM/r$, but then (2.8) and therefore all the preceding inequalities as well will be fulfilled for $r > c\sqrt{2}$. Since c = 210 km, the last inequality will be fulfilled for all real earth satellites (r > 6370 km). Thus, all real ellipsoidal motions of earth satellites are stable with respect to the semi-axis and the eccentricity of the ellipsoid.

3. Stability of hyperboloidal orbits. Let us consider the particular solution of the system - first in (1.5) and (2.1)

$$u = u_0, \qquad u' = 0, \qquad h = h_0, \qquad c_1 = c_{10} \tag{3.1}$$

This solution will exist if u_0 is a root of the equation

$$\frac{fM\sigma}{c^3}\sin u_0 + \frac{c_{10}^2\cos u_0}{\sin^3 u_0} - \frac{h_0}{c^2}\sin 2u_0 = 0$$

In this case the point will move on the surface of the hyperboloid

$$\frac{x^2}{c^2\sin^2 u_0}+\frac{y^2}{c^2\sin^2 u_0}-\frac{(z-c\zeta)^2}{c^2\cos^2 u_0}=1$$

Its real and imaginary semi-axes will be

$$a_1=c\sin u_0, \qquad b_1=c\cos u_0 \tag{3.2}$$

In particular, for $c_{10} = 0$ the point will move on a certain meridional hyperbola. Let us denote the values of the variables in perturbed motion by

$$u=u_0+x_1, \qquad u'=x_2, \qquad h=h_0+x_3, \qquad c_1^2=c_{10}^2+x_4$$

Then the differential equations of perturbed motion

$$\frac{dx_1}{d\tau}=x_2, \qquad \frac{dx_3}{d\tau}=0, \qquad \frac{dx_4}{d\tau}=0$$

$$\frac{dx_2}{d\tau}=\frac{fM\sigma}{c^3}\sin(u_0+x_1)+\frac{(c_{10}^2+x_4)\cos(u_0+x_1)}{\sin^3(u_0+x_1)}-\frac{h_0+x_3}{c^2}\sin 2(u_0+x_1)$$

possess the first integrals

$$F_1=x_2^2+\frac{2fM\sigma}{c^3}\cos(u_0+x_1)+\frac{c_{10}^2+x_4}{\sin^2(u_0+x_1)}-\frac{2fM\sigma}{c^3}\cos u_0-$$

$$-\frac{c_{10}^2}{\sin^2 u_0}+\frac{h_0}{c^2}\cos 2u_0=\text{const}, \qquad F_2=x_3=\text{const}, \qquad F_3=x_4=\text{const} \tag{3.3}$$

In order to prove the stability of the unperturbed motion (3.1) we construct the Liapunov function, according to Chetaev [4], in the form of a combination of integrals

$$W=F_1+\frac{\cos 2u_0}{c^2}F_2-\frac{1}{\sin^2 u_0}F_3+\mu_2F_2^2+\mu_3F_3^2$$

Here μ_2 and μ_3 are arbitrary constants.

Expanding W into a Taylor series in the neighborhood of $x_1 = x_2 = x_3 = x_4 = 0$, and retaining terms up to the second order, we obtain

$$W=\mu_1x_1^2+x_2^2+\mu_2x_3^2+\mu_3x_4^2+\frac{2}{c^2}\sin 2u_0x_1x_3-\frac{2\cos u_0}{\sin^3 u_0}x_1x_4+\ldots$$

Since in view of (3.3), the differential of W is equal to zero, then for stability of the unperturbed motion (3.1) it is sufficient, according to a Liapunov theorem [5], that the following inequalities be fulfilled.

$$\mu_1=\frac{4c_{10}^2\cos u_0}{\sin^4 u_0}-\frac{fM\sigma\sin^2 u_0}{\cos u_0}>0 \tag{3.4}$$

$$\mu_2\mu_1-\frac{\sin^2 2u_0}{c^4}>0, \qquad \mu_3\left(\mu_2\mu_1-\frac{\sin^2 2u_0}{c^4}\right)-\mu_2\frac{\cos^2 u_0}{\sin^6 u_0}>0 \tag{3.5}$$

Obviously, if the inequality (3.4) is fulfilled then μ_2 and μ_3 can be found such that the inequality (3.5) is also fulfilled. Taking (3.2) into account, the inequality (3.4) can be rewritten as

$$\frac{4c_{10}^2 b_1 c^3}{a_1^4} > \frac{fM\sigma a_1^2}{b_1 c}$$

which will be fulfilled since $c > 0$, $b_1 > 0$, $\sigma < 0$.

Thus, the hyperboloidal motions are stable with respect to the semi-axes and the eccentricity of the hyperboloid.

4. Stability of circular orbits. If $u = u_0$, $v = v_0$, then the point moves on a circle

$$x^2 + y^2 = c^2 \cosh^2 v_0 \sin^2 u_0, \qquad z = c\sigma - c \sinh v_0 \cos u_0$$

In this case the point is located on the surface of the ellipsoid and the hyperboloid, and therefore all real motions of earth satellites are stable with respect to the radius of the circle.

A special investigation is required for the equatorial circular orbits ($z = c\sigma$) since then $\cos u_0 = 0$ and the inequality (3.4) becomes meaningless. In order to investigate the stability of the equatorial circular orbits let us rewrite the potential (1.1) in terms of cylindrical coordinates ρ, ψ, z

$$U = \frac{fM}{2}\left[\frac{1 + i\sigma}{\sqrt{\rho^2 + [z - c(\sigma + i)]^2}} + \frac{1 - i\sigma}{\sqrt{\rho^2 + [z - c(\sigma - i)]^2}}\right]$$

It is shown in [6] that the circular motions of the considered equatorial type $\rho = \rho_0$, $z = c\sigma$ will be stable if and only if certain inequalities are fulfilled. In this case these inequalities are of the form

$$\frac{fM\rho_0}{\sqrt{(\rho_0^2 + c^2)^5}} > 0, \qquad \frac{fM(\rho_0^2 - 3c^2)}{\sqrt{(\rho_0^2 + c^2)^5}} > 0$$

It can be seen from this that the inequalities will be fulfilled for $\rho_0 > c\sqrt{3}$, i.e. for all real satellites of the earth.

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